Tutorial 3

1. (i) Markov process but not a martingale

$$E_n[f(S_{n+1})] = pf(uS_n) + qf(dS_n) = g(S_n)$$

where g is set to be $g(x) = pf(ux) + qf(dx)$.

$\{S_n\}_{n\geq 0}$ is Markov process.

However, $E_n[S_{n+1}] = puS_n + qdS_n = (pu+qd)S_n$

unless $pu + qd = 1$ otherwise $\{S_n\}_{n\geq 1}$ is not martingale.

(ii) Martingale but not a Markov process

The process $Y_0 = S_0$, $Y_N = S_0 + S_1 + \dots + S_N$, and $Z_n = E_n[Y_N]$

For $N \geq m \geq n$, $E_n[Z_m] = E_n[E_m[Y_N]] = E_n[Z_N] = Z_n$.

$\{Z_n\}_{n\geq 0}$ is martingale.

However, if we take $N=2$, then $Z_2 - Z_1 = S_0 + S_1 + S_2 - E_1(S_0 + S_1 + S_2)$

$= S_2 - E_1(S_2)$

$E_1(Z_2 - Z_1)^2 = E_1(S_2 - E_1(S_2))^2 = E_1(S_2 - S_1(pu+qd))^2 = S_1^2[p(u-(pu+qd))^2 + q(d-(pu+qd))]^2$

$\overset{\text{set as}}{=} f(S_1)$.

Also, $E_1 Z_2^2 - Z_1^2 = E_1(Z_2^2 - Z_1 Z_2) + E_1 Z_1(Z_2 - Z_1) = E_1(Z_2 - Z_1)(Z_2 - Z_1) + 2E_1 Z_1(Z_2 - Z_1)$

$= E_1(Z_2 - Z_1)^2 + 2Z_1 E_1(Z_2 - Z_1) = E_1(Z_2 - Z_1)^2$

then $E_1[Z_2^2] = Z_1^2 + f(S_1) \neq g(Z_1)$ If $Z_1 = \tilde{f}(S_1)$, $Z_1 = S_0 + S_1 + E_1[S_2]$

$\tilde{f}(S_1) - h(S_1) = S_0 \underset{\text{contradiction}}{\Leftarrow} = S_0 + h(S_1)$.

2. $p=q=\frac{1}{2}$, $X_j=\pm 1$, $M_0=0$, $M_n=\sum_{j=1}^{n} X_j$, $I_n=\sum_{j=1}^{n-1} M_j(M_{j+1}-M_j)$

$$2I_n = 2\sum_{j=0}^{n-1} M_j(M_{j+1}-M_j) = 2\sum_{j=0}^{n-1} M_jM_{j+1} - \sum_{j=1}^{n-1} M_j^2 - \sum_{j=1}^{n-1} M_j^2$$

$$= 2\sum_{j=0}^{n-1} M_jM_{j+1} + M_n^2 - \sum_{j=0}^{n-1} M_{j+1}^2 - \sum_{j=0}^{n-1} M_j^2 = M_n^2 - \sum_{j=0}^{n-1}(M_{j+1}-M_j)^2$$

$$= M_n^2 - \sum_{j=0}^{n-1} X_{j+1}^2 = M_n^2 - n$$

then $|M_n| = \sqrt{2I_n+n} \overset{d}{=} h(I_n)$. also

$$I_{n+1}-I_n = M_n(M_{n+1}-M_n) = M_nX_{n+1} \Rightarrow I_{n+1} = I_n + M_nX_{n+1}$$

$$E_n[f(I_{n+1})] = E_n[f(I_n+M_nX_{n+1})] = \frac{1}{2}f(I_n+M_n) + \frac{1}{2}f(I_n-M_n)$$

$$= \frac{1}{2}f(I_n+h(I_n)) + \frac{1}{2}f(I_n-h(I_n)) = g(I_n).$$

I_n is Markov process.

3. 2.9(i) $u_0=2$, $d_0=\frac{1}{2}$, $u_1(H)=1.5$, $u_1(T)=1$, $d_1(H)=1$, $d_1(T)=1$

$\tilde{p}_0 = \frac{1+r_0-d_0}{u_0-d_0} = \frac{1}{2}$, $\tilde{q}_0=\frac{1}{2}$. similarly $\tilde{p}_1(H)=\frac{1}{2}$, $\tilde{q}_1(H)=\frac{1}{2}$,

$\tilde{p}_1(T)=\frac{1}{6}$, $\tilde{q}_1(T)=\frac{5}{6}$. Therefore, $\tilde{p}(HH)=\tilde{p}_0\tilde{p}_1(H)=\frac{1}{4}$, $\tilde{p}(HT)=\frac{1}{4}$,

$\tilde{p}(TH)=\frac{1}{12}$, $\tilde{p}(TT)=\frac{5}{12}$.

(ii) $V_2(HH)=5$, $V_2(HT)=1$, $V_2(TH)=1$, $V_2(TT)=0$. then

$V_1(H) = \frac{\tilde{p}_1(T)V_2(TH)+\tilde{q}_1(T)V_2(TT)}{1+r_1(T)} = 2.4$, $V_1(T)=\frac{1}{9}$ and $V_0=\underline{1}$.

2.14.

(i) For any $n \le M$, $(S_n, Y_n) = (S_n, 0)$. which is Markov under risk-neutral measure

For $n \ge M+1$, we have $\tilde{E}_n[g(S_{n+1}, Y_{n+1})] = \tilde{p}g(uS_n, Y_n+uS_n) + \tilde{q}(dS_n+Y_n+dS_n)$
$= h(S_n, Y_n)$. where

$h(s,y) = \tilde{p}g(us, y+us) + \tilde{q}g(ds, y+ds)$.

(ii)

(a) If $n > M$, then $\tilde{E}_n[V_{n+1}(S_{n+1}, Y_{n+1})] = V_n(S_{n+1}, Y_{n+1})$

where $V_n(S, y) = \tilde{p} V_{n+1}(uS, y+ds) + \tilde{q} V_{n+1}(ds, y+ds)$.

(b) If $n = M$, then $\tilde{E}_n[V_{M+1}(S_{M+1}, Y_{M+1})] = V_M(S_M)$.

and $V_M(S) = \tilde{p} V_{M+1}(uS, uS) + \tilde{q} V_{M+1}(ds, ds)$

(c) If $n < M$, then $\tilde{E}_n[V_{n+1}(S_{n+1})] = V_n(S_n)$, where

$V_n(S) = \tilde{p} V_{n+1}(uS) + \tilde{q} V_{n+1}(ds)$.

Ex 2.5.

(i) Proved by Question 2.

(ii) Proved by Question 2.

Ex 3.3

$M_2(HH) = 32 \times \frac{2}{3} + 8 \times \frac{1}{3} = 24$, $M_2(HT) = 8 \times \frac{2}{3} + 2 \times \frac{1}{3} = 6$,

$M_2(TH) = 6$, $M_2(TT) = 1.5$. $M_1(H) = 24 \times \frac{2}{3} + 6 \times \frac{1}{3} = 18$, $M_1(T) = 4.5$

$M_0 = \frac{2}{3} \times 18 + \frac{1}{3} \times 4.5 = 12 + 1.5 = 13.5$.

We can see that $\tilde{E}_n[M_{n+1}] = \tilde{E}_n[\tilde{E}_{n+1}[S_3]] = \tilde{E}_n[S_3] = M_n$.

$n \le 3$.

It is martingale.

Ex 2.6

$\tilde{E}_n[I_{n+1} - I_n] = \tilde{E}_n[\Delta_n(M_{n+1} - M_n)] = \Delta_n \tilde{E}_n[M_{n+1} - M_n] = 0$.

then $\tilde{E}_n[I_{n+1}] = I_n$. $n \ge 0$.